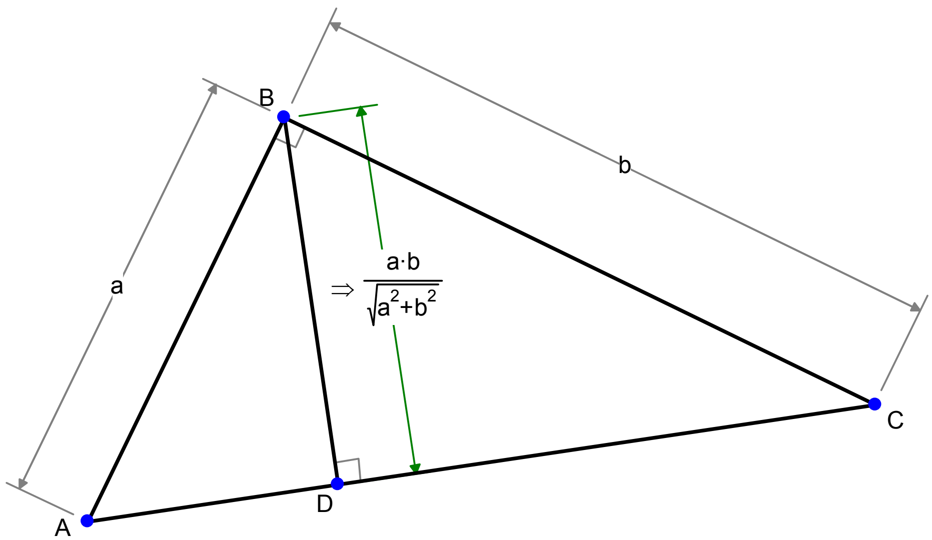
# Mechanisms, Splines and Caustics with Geometry Expressions

# Introduction

***Geometry Expressions*** automatically generates algebraic expressions from geometric figures. For example in the diagram below, the user has specified that the triangle is right and has short sides length a and b. The system has calculated an expression for the length of the altitude:




We present a collection of worked examples using Geometry Expressions. In most cases, a diagram is presented with little comment. It is hoped that these diagrams are sufficiently self explanatory that the reader will be able to create them himself.

The goal of these examples is to demonstrate the sort of problems which the software is capable of handling, and to suggest avenues of further exploration for the reader.

The examples are clustered by theme.

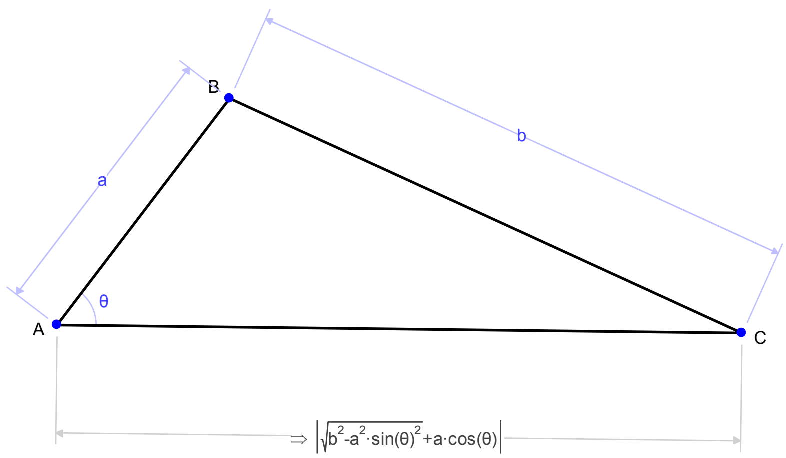
## Mechanisms

Geometry Expressions provides an excellent environment for defining mechanisms. Specifying a length constraints corresponding to rigid members of the mechanism, specify coordinate constraints corresponding to grounded points. Specify angles to correspond to motors.

There follow a few mechanism examples.

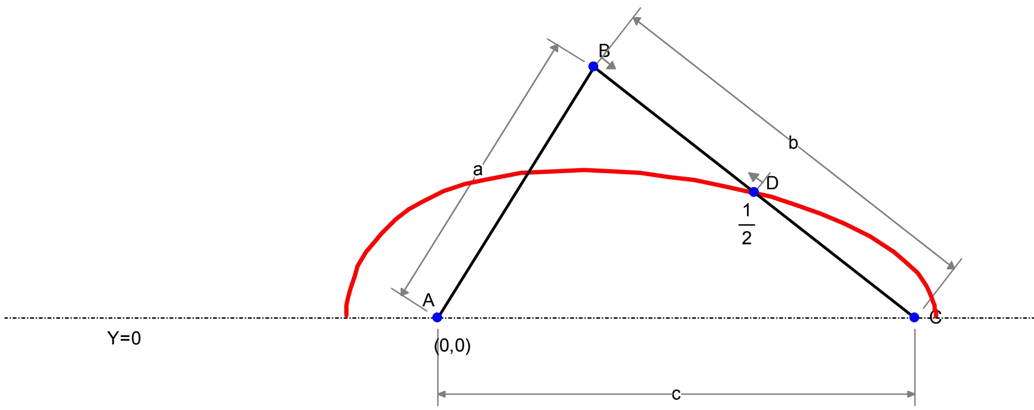
## Example 1: A Crank Piston Mechanism

For crank length c and connecting rod length L, we compute piston displacement as a function of angle



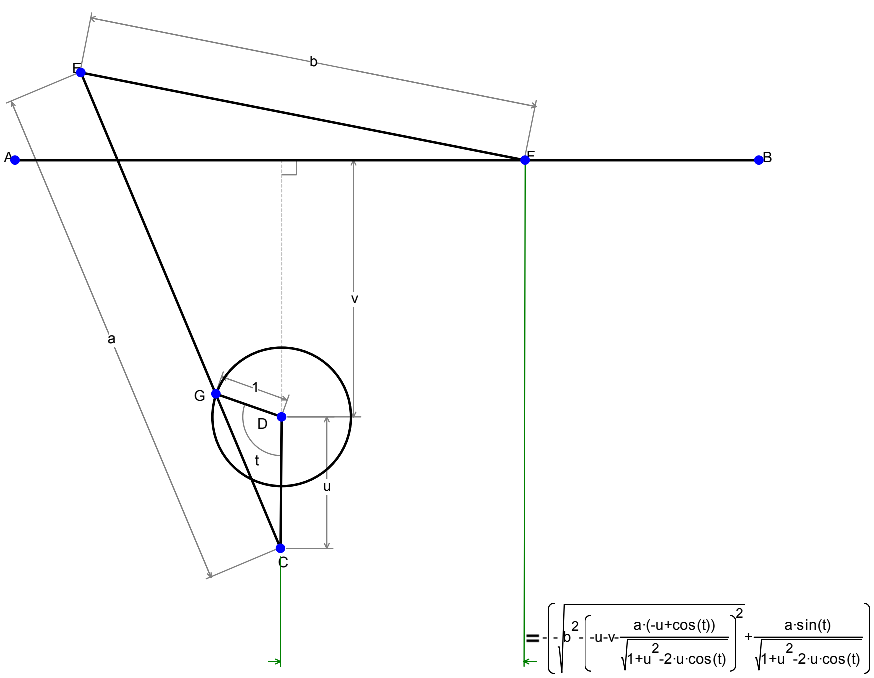
## Example 2: A Crank Slider Coupler Curve

$$\Rightarrow 16\cdot X^4+160\cdot X^2\cdot Y^2+144\cdot Y^4+16\cdot a^4-8\cdot a^2\cdot b^2+b^4+Y^2\cdot\left(-96\cdot a^2+24\cdot b^2\right)+X^2\cdot\left(-32\cdot a^2-8\cdot b^2\right)=0$$



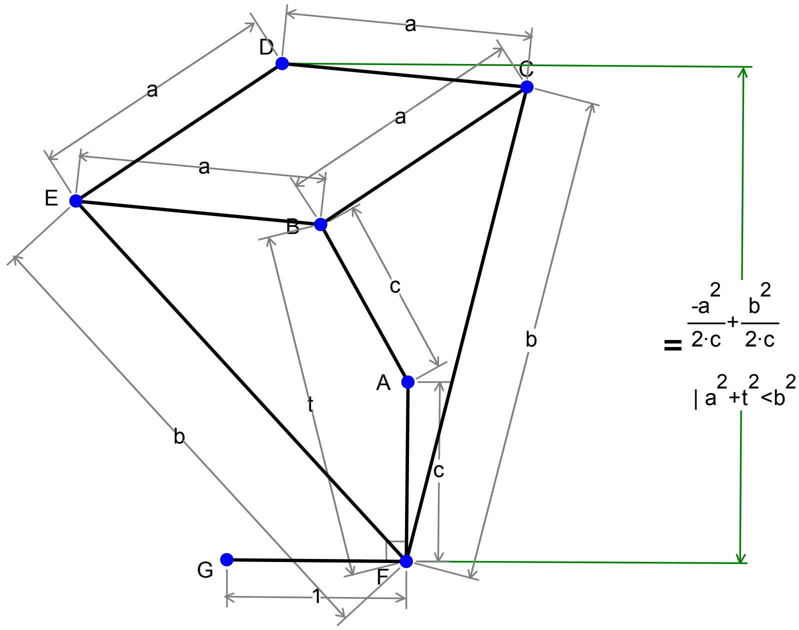
## Example 3: A Quick Return Mechanism

The crank DG operates a quick return mechanism whose end-effector is at point F. The formula shows the horizontal displacement of F in terms of t and the various parameters of the geometry: a,b,u,v:



## Example 4: Paucellier's Linkage

In Paucellier's linkage, we look at the height of the end-effector:

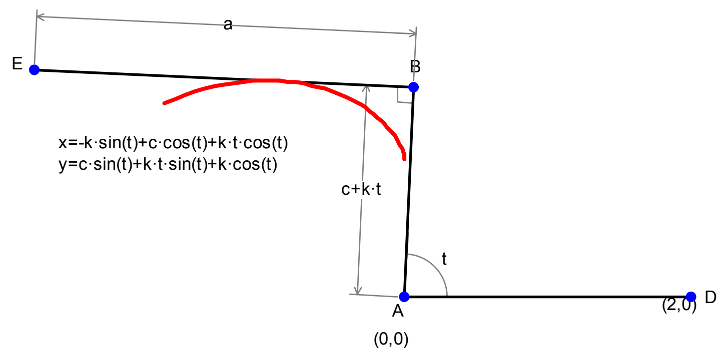



We see this is invariant in t.

The mechanism was the first to convert purely rotary motion (of the crank AB) to exact linear motion of the end effector D.

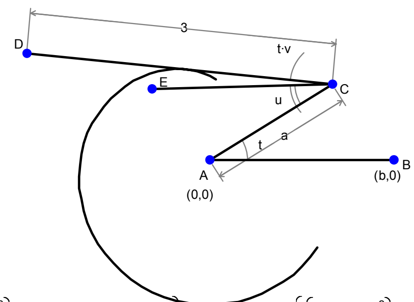
## Example 5: Steady Rise Cam Curve

Assuming a Flat Plate reciprocating follower, here is the cam curve for a linear rise of k*t+c. This is the Envelope of the line BE.



## Example 6: Oscillating Flat Plate Cam

Here is a cam curve for an oscillating flat plate cam follower, where the follower rise is linear in the cam angle: rise = u+t*v

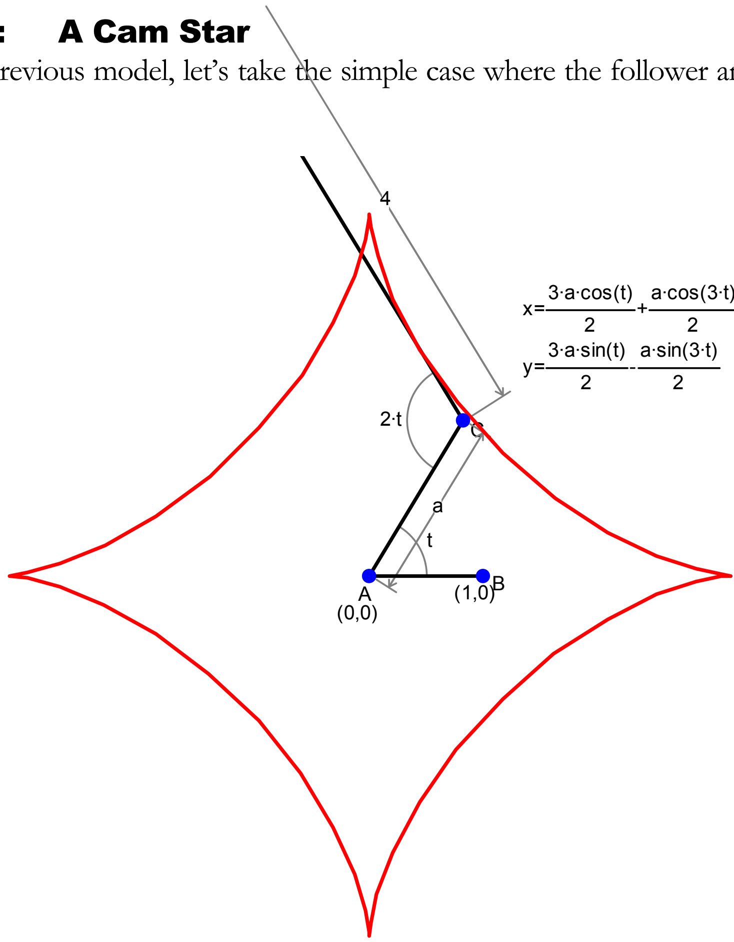



$$x=\frac{a\cdot\left(\left(-1+2\cdot\cos(t\cdot v)^{2}\right)\cdot\left(\left(-1+2\cdot\cos(u)^{2}\right)\cdot\cos(t)+2\cdot\sin(t)\cdot\sin(u)\cdot\cos(u)\right)-\cos(t)+2\cdot v\cdot\cos(t)-2\cdot\left(-\left(-1+2\cdot\cos(u)^{2}\right)\cdot\sin(t)+2\cdot\sin(u)\cdot\cos(t)\cdot\cos(u)\right)\cdot\sin(t\cdot v)\cdot\cos(t\cdot v)\right)}{2\cdot(-1+v)}\ \Big|\ b>0$$

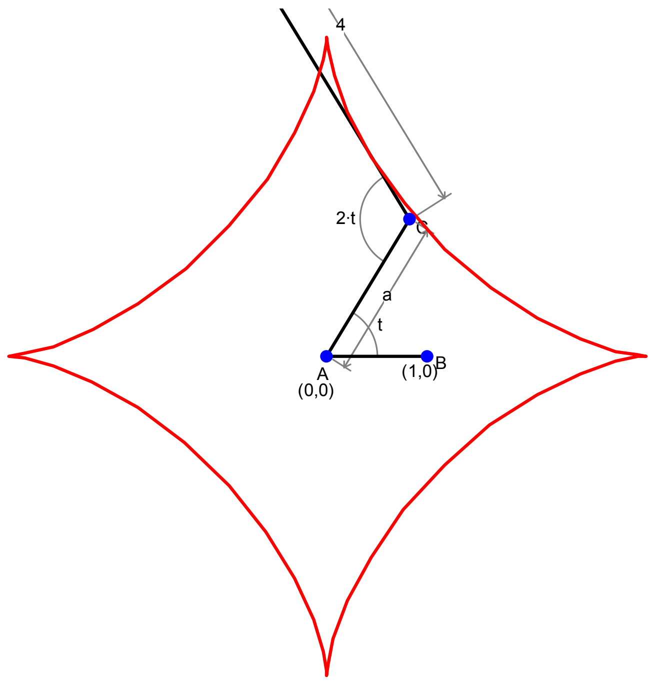
$$y=\frac{a\cdot\left(-\left(\left(-1+2\cdot\cos(t\cdot v)^{2}\right)\cdot\left(-\left(-1+2\cdot\cos(u)^{2}\right)\cdot\sin(t)+2\cdot\sin(u)\cdot\cos(t)\cdot\cos(u)\right)+2\cdot\left(\left(-1+2\cdot\cos(u)^{2}\right)\cdot\cos(t)+2\cdot\sin(t)\cdot\sin(u)\cdot\cos(u)\right)\cdot\sin(t\cdot v)\cdot\cos(t\cdot v)\right)-\sin(t)+2\cdot v\cdot\sin(t)\right)}{2\cdot(-1+v)}\ \Big|\ b>0$$

## Example 7: A Cam Star

Based on the previous model, let's take the simple case where the follower angle is twice the cam angle:



Can we get an implicit definition of the curve?  Yes.




$$-64\cdot a^6+x^6+48\cdot a^4\cdot y^2-12\cdot a^2\cdot y^4+y^6+x^4\cdot\left(-12\cdot a^2+3\cdot y^2\right)+x^2\cdot\left(48\cdot a^4+84\cdot a^2\cdot y^2+3\cdot y^4\right)=0$$

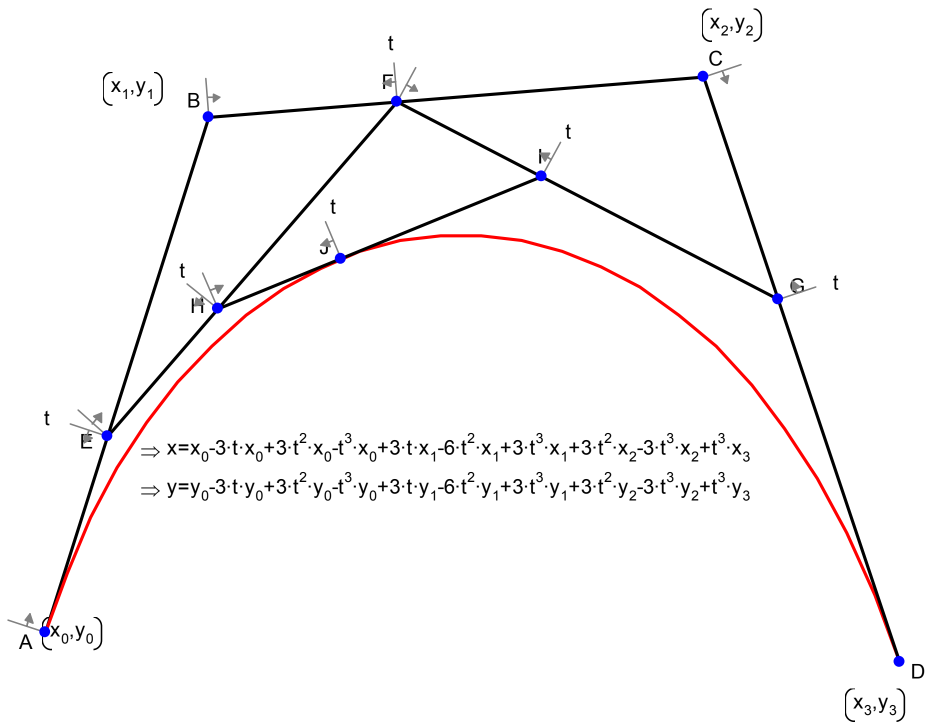
# Spline curves

Splines are curve families which are typically used in describing free form geometry in computer aided design environments. Geometry Expressions lets us explore the mathematical properties of some of the curves.

Here are a few examples

## Example 8: Cubic Spline

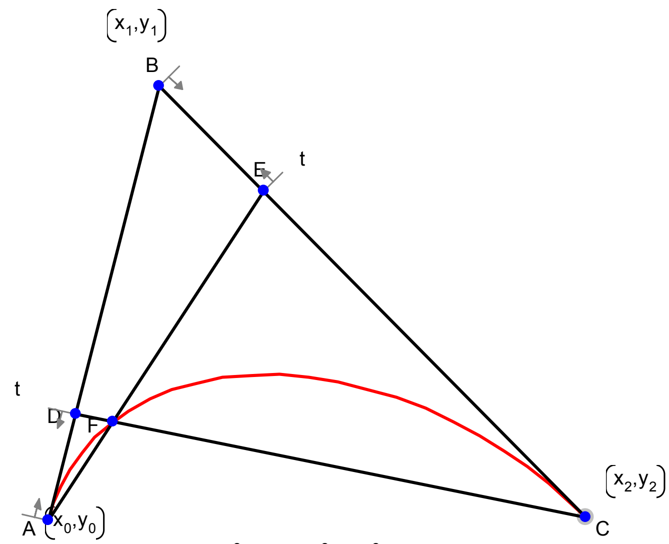
This diagram shows an algorithm for constructing the cubic spline from its control points:




Point E is proportion t along the line AB. Point F is proportion t along BC. Point G is proportion t along CD. Point H is proportion t along EF. Point I is proportion t along FG. Point J is proportion t along HI. The spline curve is the locus as t runs from 0 to 1.

## Example 9: A Triangle Spline

We can create another spline curve from 3 control points ABC in the following way: Point D is located proportion t along AB. Point E is located proportion t along BC. We take the locus of the intersection of AE and CD:




$$\Rightarrow x=\frac{x_0-2\cdot t\cdot x_0+t^2\cdot x_0+t\cdot x_1-t^2\cdot x_1+t^2\cdot x_2}{1-t+t^2}$$

$$\Rightarrow y=\frac{y_0-2\cdot t\cdot y_0+t^2\cdot y_0+t\cdot y_1-t^2\cdot y_1+t^2\cdot y_2}{1-t+t^2}$$

Copy the x coordinate into Maple and differentiate to get:

```
> u:= diff((x[`0`]-2*x[`0`]*t+x[`0`]*t^2+x[`1`]*t-
x[`1`]*t^2+x[`2`]*t^2)/(-t+1+t^2),t);
```

$$u := \frac{-2\,x_0 + 2\,x_0\,t + x_1 - 2\,x_1\,t + 2\,x_2\,t}{-t + 1 + t^2} - \frac{(x_0 - 2\,x_0\,t + x_0\,t^2 + x_1\,t - x_1\,t^2 + x_2\,t^2)\,(-1 + 2\,t)}{(-t + 1 + t^2)^2}$$

Substituting t=0 and t=1::

```
> subs(t=0,u);
```

$$-x_0 + x_1$$

```
> subs(t=1,u);
```

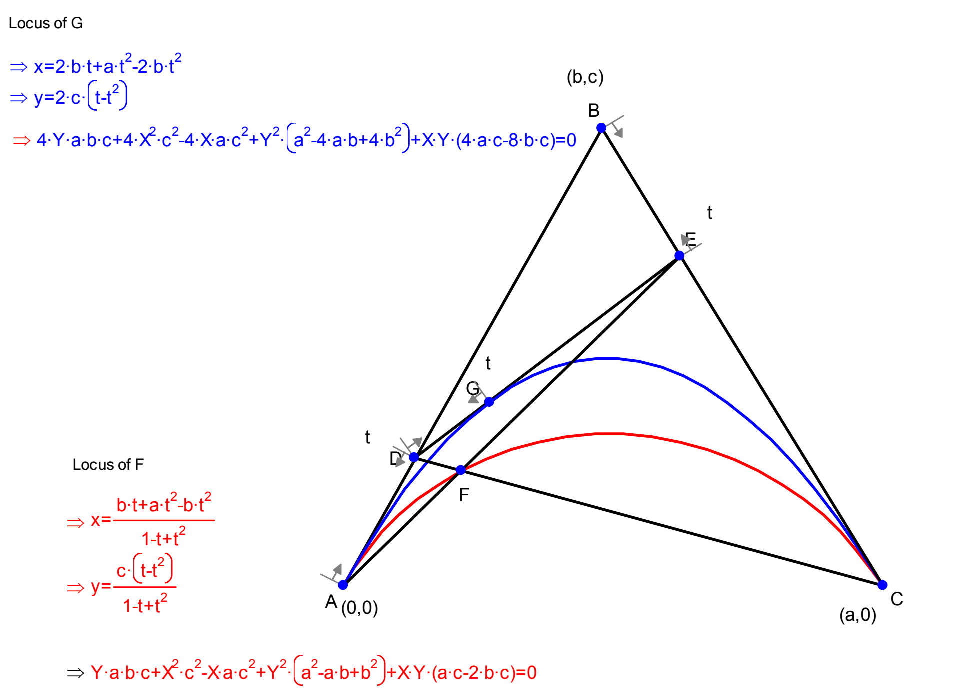
$$-x_1 + x_2$$

Comparable result for y shows that the curve is tangent to the control triangle at the end points

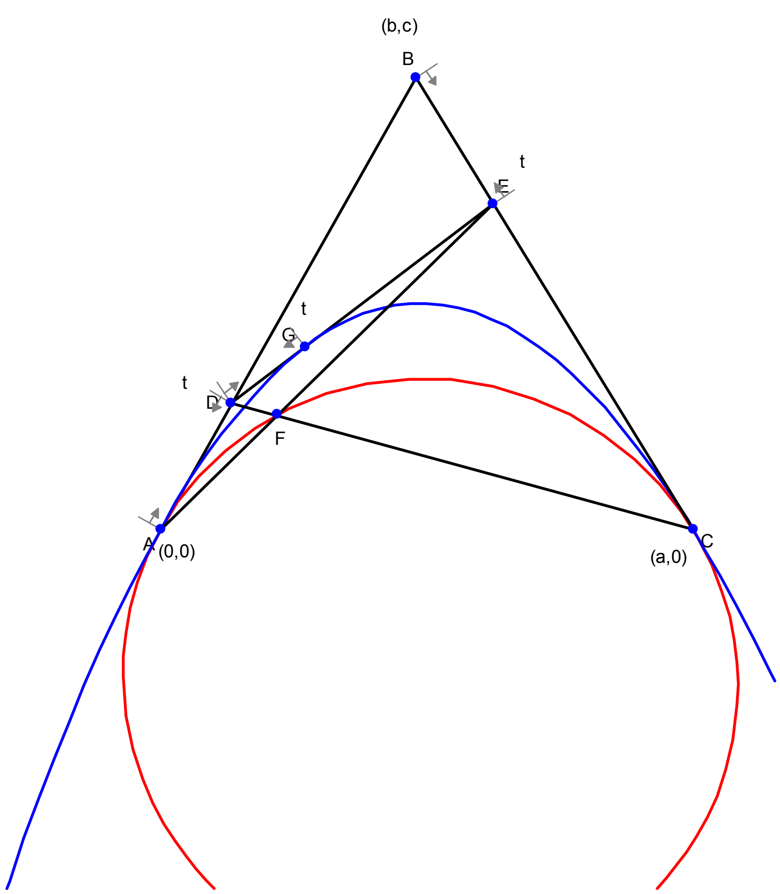
## Example 10: Another Triangle Spline

We can also create a spline from a control triangle by taking the locus of a point G proportion t along DE.

Observing the parametric form of the curves we see that one is a parametric quadratic, while the other is a rational quadratic. Implicit forms are both conics (and almost, but not quite, identical).



What types of conics are they? Extending the curves a little can give a clue:




The blue curve looks like a parabola, the red certainly does not.

Copying the blue curve equation into Maple and examining the quadratic form shows that it is indeed a parabola:

```
> 4*c*b*a*Y+4*c^2*X^2-4*c^2*a*X+(a^2-
4*b*a+4*b^2)*Y^2+(4*c*a-8*c*b)*Y*X = 0;
```

$$4\,c\,b\,a\,Y + 4\,c^2\,X^2 - 4\,c^2\,a\,X + (a^2 - 4\,b\,a + 4\,b^2)\,Y^2 + (4\,c\,a - 8\,c\,b)\,Y\,X = 0$$

```
> <<4*c^2 |(4*c*a-8*c*b)/2>, <(4*c*a-8*c*b)/2 |(a^2-4*b*a+4*b^2)>>;
```

$$\begin{bmatrix} 4\,c^2 & 2\,c\,a-4\,c\,b \\ 2\,c\,a-4\,c\,b & a^2-4\,b\,a+4\,b^2 \end{bmatrix}$$

```
> Determinant(%);
```

$$0$$

How about the red curve:?

```
> c*b*a*Y+c^2*X^2-c^2*a*X+(a^2-b*a+b^2)*Y^2+(c*a-2*c*b)*Y*X = 0;
```

$$c\,b\,a\,Y+c^2\,X^2-c^2\,a\,X+(a^2-b\,a+b^2)\,Y^2+(c\,a-2\,c\,b)\,Y\,X=0$$

```
> <<c^2 |(c*a-2*c*b)/2>, <(c*a-2*c*b)/2 |(a^2-b*a+b^2)>>;
```

$$\begin{bmatrix} c^2 & \frac{1}{2}\,c\,a-c\,b \\ \frac{1}{2}\,c\,a-c\,b & a^2-b\,a+b^2 \end{bmatrix}$$

```
> Determinant(%);
```

$$\frac{3}{4}\,c^2\,a^2$$

We see that the determinant is positive. This means we will always have a portion of an ellipse, never a hyperbola (hyperbolas can be undesireable curves, as one rarely wants an asymptote in the middle of whatever curve one is working with).

## Caustics

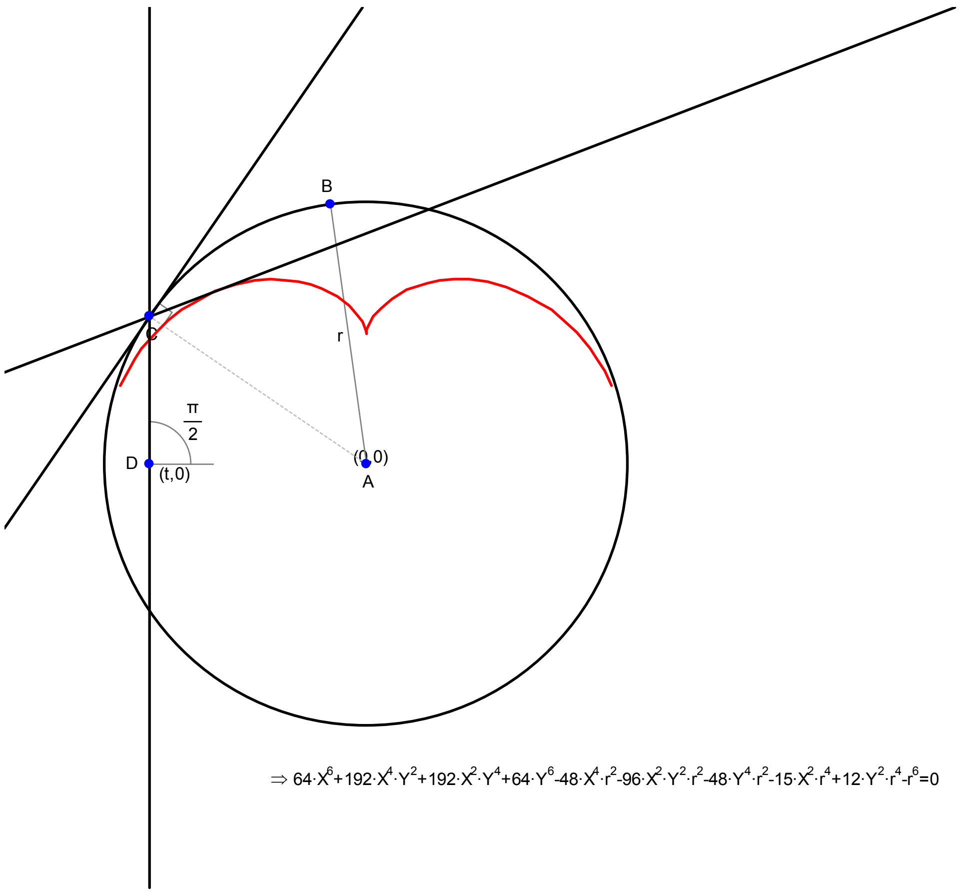
A caustic is the light curve generated when the reflection of a bundle rays align themselves along a specific curve.

Mathematically, it is the envelope of the reflected family of rays.

Here are a couple of examples.

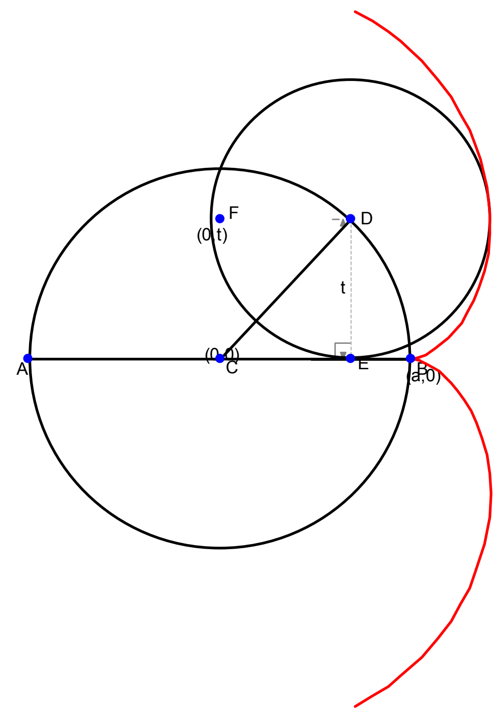
## Example 11: Caustics in a cup of coffee

The Nephroid curve generated by reflecting a set of parallel rays in a circle, and then taking the envelope of the reflected rays:



## Example 12: A Nephroid by another route

The envelope of the circles whose centers lie on a circle and which are tangential to the diameter form the same type of curve:




$$\Rightarrow 4\cdot X^6+12\cdot X^4\cdot Y^2+12\cdot X^2\cdot Y^4+4\cdot Y^6-12\cdot X^4\cdot a^2-24\cdot X^2\cdot Y^2\cdot a^2-12\cdot Y^4\cdot a^2+12\cdot X^2\cdot a^4-15\cdot Y^2\cdot a^4-4\cdot a^6=0$$

## Example 13: Tschirnhausen's Cubic

Studied by Ehrenfried Tschirnhausen in 1690, this is the caustic of a set of parallel rays perpendicular to the axis of a parabola: